

The Shintani descents of Suzuki Groups and consequences

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ABSTRACT

The main aim of this paper is to associate to every cuspidal unipotent character of the Suzuki group its root of unity and to give a possible definition of the Fourier matrix associated to the family of the cuspidal unipotent characters of this group. We compute to this end the Shintani descents of Suzuki groups and use results of Digne and Michel.

1 Introduction

Let \mathbf{G} be a connected reductive group defined over the finite field with q elements and let $F : \mathbf{G} \rightarrow \mathbf{G}$ be a generalized Frobenius map. Let W be the Weyl group of \mathbf{G} which respect to an F -stable maximal torus of \mathbf{G} contained in an F -stable Borel subgroup of \mathbf{G} . We denote by \mathbf{G}^F the finite group of fixed points under F . For $w \in W$ we have a corresponding generalized Deligne-Lusztig character R_w of \mathbf{G}^F . We denote by $\mathcal{U}(\mathbf{G}^F)$ the set of unipotent characters of \mathbf{G}^F , that is the irreducible constituent of the R_w for $w \in W$. In [9] Lusztig attached to every $\chi \in \mathcal{U}(\mathbf{G}^F)$ a root of unity ω_χ . In [7] he computes all such roots of unity corresponding to unipotent characters of finite reductive groups, except for some pairs of complex conjugate characters where a sign is missing. For example, if $\mathbf{G}^F = \text{Sz}(2^{2n+1})$ (where n is a non-negative integer) is the Suzuki group with parameter 2^{2n+1} then \mathbf{G}^F has two cuspidal unipotent complex conjugate characters \mathcal{W} and $\overline{\mathcal{W}}$ of degree $2^n(2^{2n+1} - 1)$; Lusztig's method only gives that $\omega_{\mathcal{W}} = \frac{\sqrt{2}}{2}(-1 \pm \sqrt{-1})$. On the other hand, using the *almost characters* of \mathbf{G}^F , Lusztig has shown that the unipotent characters of \mathbf{G}^F can be distributed in families. In [9] Lusztig associated to most of the families a matrix, the so-called Fourier matrix of the family. However the Suzuki and Ree groups do not have Fourier matrices in the Lusztig sense! On the other side, Geck and Malle have axiomatized Fourier matrices and give in [11] candidates for these groups.

The aim of this paper is to compute the roots of unity attached to the cuspidal unipotent characters \mathcal{W} and $\overline{\mathcal{W}}$ of the Suzuki group with parameter 2^{2n+1} and to give a possible definition of the Fourier matrix associated to the family $\{\mathcal{W}, \overline{\mathcal{W}}\}$ with another approach as the one proposed by Geck and Malle. To this end, we compute *the Shintani descents* of the Suzuki group and use results of Digne-Michel [5].

Let n be a non-negative integer and let \mathbf{G} be a simple group of type B_2 defined over $\overline{\mathbb{F}}_2$. Let F be the generalized Frobenius map such that \mathbf{G}^F is the Suzuki group with parameter 2^{2n+1} . The finite group $\mathbf{G}^{F^2} = B_2(2^{2n+1})$ is the finite "untwisted" group of type B_2 with parameter 2^{2n+1} .

This paper is organized as follows: in the first section, we recall some definitions and generalities. In the second section, we explicitly compute the character table of the finite group $B_2(2^{2n+1}) \rtimes \langle \sigma \rangle$, where σ is the restriction of F to $B_2(2^{2n+1})$. The main result of this part is:

Theorem 1.1 *Let n be a non-negative integer. We set $q = 2^{2n+1}$ and let σ the exceptional graph automorphism of $B_2(q)$ such that its fixed points subgroup is the Suzuki group with parameter q . Then the group $B_2(q) \rtimes \langle \sigma \rangle$ has $(2q + 6)$ irreducible extensions of $(q + 3)$ irreducible σ -stable characters of $B_2(q)$. The values of these extensions are given in Table 9.*

In the last section, we compute the Shintani descents of the Suzuki group with parameter 2^{2n+1} . We then obtain two consequences on the unipotent characters of the Suzuki group: we first explicitly compute the root of unity associated to their unipotent characters and secondly we compute a Fourier matrix for the families of these groups.

I wish to express my hearty thanks to Meinolf Geck for leading me to this work and for valuable discussions.

2 Generalities

2.1 Finite reductive groups

Let \mathbf{G} be a connected reductive group defined over the finite field with $q = p^f$ elements. Let F be a generalized Frobenius map over \mathbf{G} . We recall that the finite subgroup $\mathbf{G}^F = \{x \in \mathbf{G} \mid F(x) = x\}$ is a so-called *finite reductive group*. Let \mathbf{H} be an F -stable maximal torus of \mathbf{G} contained in an F -stable Borel \mathbf{B} of \mathbf{G} . We set $W = N_{\mathbf{G}}(\mathbf{H})/\mathbf{H}$ the Weyl group of \mathbf{G} . The map F induces an automorphism of W (also denoted by F to simplify). We denote by δ the order of this automorphism.

We fix $w \in W$ and we define the corresponding Deligne-Lusztig variety by:

$$X_w = \{x\mathbf{B} \mid x^{-1}F(x) \in \mathbf{B}w\mathbf{B}\}.$$

We recall that for every positive integer i , we can associate a $\overline{\mathbb{Q}}_\ell$ -space to X_w , the i -th ℓ -adic cohomology space with compact support $H_c^i(X_w, \overline{\mathbb{Q}}_\ell)$ over the

algebraic closure $\overline{\mathbb{Q}_\ell}$ of the ℓ -adic field (Here, ℓ is a prime not dividing q). The group \mathbf{G}^F acts on X_w . This action induces a linear action on $H_c^i(X_w, \overline{\mathbb{Q}_\ell})$. Thus these spaces are $\overline{\mathbb{Q}_\ell}\mathbf{G}^F$ -modules. We define the generalized Deligne-Lusztig character by:

$$\forall g \in \mathbf{G}^F, \quad R_w(g) = \sum_{i \geq 0} (-1)^i \text{Tr}(g, H_c^i(X_w, \overline{\mathbb{Q}_\ell})).$$

The set of irreducible characters of \mathbf{G}^F is denoted by $\text{Irr}(\mathbf{G}^F)$ and we denote by $\langle \cdot, \cdot \rangle_{\mathbf{G}^F}$ the usual scalar product on the space $\text{C}(\mathbf{G}^F)$ of the $\overline{\mathbb{Q}_\ell}$ -valued class functions of \mathbf{G}^F . We define the set $\mathcal{U}(\mathbf{G}^F)$ of unipotent characters of \mathbf{G}^F by:

$$\mathcal{U}(\mathbf{G}^F) = \{\chi \in \text{Irr}(\mathbf{G}^F) \mid \exists w \in W, \langle R_w, \chi \rangle_{\mathbf{G}^F} \neq 0\}.$$

2.1.1 The root of a unipotent character

The group $\langle F^\delta \rangle$ acts on X_w . This action induces a linear endomorphism on $H_c^i(X_w, \overline{\mathbb{Q}_\ell})$. We also fix an eigenvalue λ of F^δ on $H_c^i(X_w, \overline{\mathbb{Q}_\ell})$ and we denote by $F_{\lambda,i}$ its generalized eigenspace. The actions of \mathbf{G}^F and of $\langle F^\delta \rangle$ on $H_c^i(X_w, \overline{\mathbb{Q}_\ell})$ commute, thus $F_{\lambda,i}$ is a $\overline{\mathbb{Q}_\ell}\mathbf{G}^F$ -module. Moreover the irreducible constituents which occur in the character associated to this $\overline{\mathbb{Q}_\ell}\mathbf{G}^F$ -module are unipotent characters of \mathbf{G}^F . Now let $\chi \in \mathcal{U}(\mathbf{G}^F)$. Then there exists $w \in W$, $\lambda \in \overline{\mathbb{Q}_\ell}^\times$ and $i \in \mathbb{N}$ such that χ occurs in the character associated to $F_{\lambda,i}$. Lusztig has shown that λ , up to a power of $q^{1/2}$, is a root of unity which depends only on χ (denoted by ω_χ). Thus there exists $s \in \mathbb{N}$ such that $\lambda = \omega_\chi q^{s/2}$ (see [5]).

2.1.2 Fourier matrices

We assume that $\delta \neq 1$. We recall that $\langle F \rangle$ acts on $\text{Irr}(W)$. More precisely if $\rho \in \text{Irr}(W)$, we define ρ^F by $\rho^F(w) = \rho(F(w))$ for every $w \in W$. Let $\rho \in \text{Irr}(W)$ such that $\rho^F = \rho$, i.e., the inertial group of ρ in $W \rtimes \langle F \rangle$ is $W \rtimes \langle F \rangle$. It follows that ρ has extensions to $W \rtimes \langle F \rangle$. Let $\tilde{\rho}$ be such an extension; we define the *almost character* associated to $\tilde{\rho}$ by:

$$\mathcal{R}_{\tilde{\rho}} = \frac{1}{|W|} \sum_{w \in W} \tilde{\rho}(w, F) R_w,$$

where the elements of $W \rtimes \langle F \rangle$ are denoted by (w, x) for every $w \in W$ and $x \in \langle F \rangle$. Let $\chi, \chi' \in \mathcal{U}(\mathbf{G}^F)$. The characters χ and χ' are in the same family if and only if there exists $(\chi_i)_{i=1, \dots, m}$, where $\chi_i \in \mathcal{U}(\mathbf{G}^F)$ such that:

- We have $\chi_1 = \chi$ and $\chi_m = \chi'$,
- For every $2 \leq i \leq m-1$, there exists an F -stable character $\rho_i \in \text{Irr}(W)$ such that

$$\langle \chi_i, \mathcal{R}_{\tilde{\rho}_i} \rangle_{\mathbf{G}^F} \neq 0 \quad \text{and} \quad \langle \chi_{i+1}, \mathcal{R}_{\tilde{\rho}_i} \rangle_{\mathbf{G}^F} \neq 0.$$

Let \mathcal{F} be a family of unipotent characters of \mathbf{G}^F obtained in this way. Except in the cases where \mathbf{G}^F is a Suzuki group or a Ree group of type G_2 or F_4 , Lusztig has shown that we can associate a matrix $M_{\mathcal{F}}$ to \mathcal{F} . We refer to [9] for details. In the case where \mathbf{G}^F is a Suzuki group or a Ree group, Geck and Malle proposed in [11] candidates for Fourier matrices of these groups in agreement with a general axiomatization of Fourier matrices that they developed.

2.1.3 Shintani descents

We recall that the Lang map associated to a generalized Frobenius map F is the map $L_F : \mathbf{G} \rightarrow \mathbf{G}, x \mapsto x^{-1}F(x)$. Since \mathbf{G}^{F^δ} is a finite F -stable subgroup and F is an automorphism of the abstract group \mathbf{G} , it follows that F restricts to an automorphism of \mathbf{G}^{F^δ} , also denoted by F . The maps F and F^δ have Lang's property, that is, their associated Lang maps are surjective (see [10] Th. 4.1.12). Using this fact, we can establish a correspondence N_{F/F^2} between \mathbf{G}^F and $\mathbf{G}^{F^\delta} \rtimes \langle F \rangle$, the so-called *Shintani correspondence*. More precisely, let $g \in \mathbf{G}^F$; by the surjectivity of the Lang map L_{F^δ} , there exists $x \in \mathbf{G}$ such that $g = L_{F^\delta}(x)$. Therefore we have $(L_F(x^{-1}), F) \in \mathbf{G}^{F^\delta} \rtimes \langle F \rangle$. This correspondence induces a bijection between the conjugacy classes of \mathbf{G}^F and the conjugacy classes of $\mathbf{G}^{F^\delta} \rtimes \langle F \rangle$ which consist of elements of the form (g, F) , with $g \in \mathbf{G}^{F^\delta}$. Moreover, we have:

$$\forall g \in \mathbf{G}^F, \quad |C_{\mathbf{G}^{F^\delta} \rtimes \langle F \rangle}(N_{F/F^2}(g))| = \delta |C_{\mathbf{G}^F}(g)|. \quad (1)$$

Using this correspondence, we can associate a class function of \mathbf{G}^F to every class function of $\mathbf{G}^{F^\delta} \rtimes \langle F \rangle$. Indeed let $\psi \in C(\mathbf{G}^{F^\delta} \rtimes \langle F \rangle)$; we then define the *Shintani descent* of ψ by $\text{Sh}_{F^\delta/F} \psi = \psi \circ N_{F/F^2}$. We refer to [5] for further details.

2.1.4 The link between Shintani descents, Roots and Fourier matrices

The set of the irreducible constituents of $\text{Ind}_{\mathbf{B}^{F^\delta}}^{\mathbf{G}^{F^\delta}} 1_{\mathbf{B}^{F^\delta}}$ is the so-called *principal series* of \mathbf{G}^{F^δ} . There exists a 1-1 correspondence between the irreducible characters of W and the characters of the principal series of \mathbf{G}^{F^δ} (see [4]). Let $\rho \in \text{Irr}(W)$, then we denote by χ_ρ its corresponding character. Similarly Malle has shown that there is a 1-1 correspondence between $\text{Irr}(W \rtimes \langle F \rangle)$ and the irreducible components of $\text{Ind}_{\mathbf{B}^{F^\delta} \rtimes \langle F \rangle}^{\mathbf{G}^{F^\delta} \rtimes \langle F \rangle} 1_{\mathbf{B}^{F^\delta}}$. We now assume that $\rho = \rho^F$, therefore ρ has irreducible extensions in $W \rtimes \langle F \rangle$. We fix such an extension $\tilde{\rho}$ and we denote by $\chi_{\tilde{\rho}}$ the corresponding character. Then $\chi_{\tilde{\rho}}$ is an extension of χ_ρ to $\mathbf{G}^{F^\delta} \rtimes \langle F \rangle$. We have:

Theorem 2.1 (*Digne-Michel [5]*) *Let $\rho \in \text{Irr}(W)$ such that $\rho^F = \rho$. Let $\tilde{\rho} \in$*

$\text{Irr}(W \rtimes \langle F \rangle)$ be an extension of ρ . Then we have:

$$\text{Sh}_{F^\delta/F} \chi_{\tilde{\rho}} = \sum_{V \in \mathcal{U}(\mathbf{G}^F)} \langle \mathcal{R}_{\tilde{\rho}}, V \rangle_{\mathbf{G}^F} \omega_V V.$$

Remark 2.1 The Theorem is proved in [5] in the case where F is a Frobenius map. But the arguments are the same when F is a generalized Frobenius map.

We now recall some conjectures of Digne and Michel (see [5]):

Conjecture 2.1 Let $\chi \in \mathcal{U}(\text{Gal}^{F^\delta})$ such that $\chi^F = \chi$. Let $\tilde{\chi}$ be an extension of χ to $\mathbf{G}^{F^\delta} \rtimes \langle F \rangle$. Then:

1. The irreducible constituents of $\text{Sh}_{F^\delta/F} \tilde{\chi}$ are unipotent characters of \mathbf{G}^F and lie in the same family \mathcal{F} ;
2. There exists a root of unity u such that

$$\pm u \text{Sh}_{F^\delta/F} \tilde{\chi} = \sum_{V \in \mathcal{F}} a_V \omega_V V.$$

In this case, the coefficients a_V give (up to a sign) a row of the Fourier matrix associated to the family \mathcal{F} .

2.2 Suzuki groups

Let \mathbf{G} be a simple group of type B_2 defined over \mathbb{F}_2 . The root system of \mathbf{G} is $\Phi = \{-a, -b, -a-b, -2a-b, a, b, a+b, 2a+b\}$, where $\Pi = \{a, b\}$ is chosen as a fundamental root system. We denote by $\Phi^+ = \{a, b, a+b, 2a+b\}$ the set of positive roots with respect to Π . The Weyl group W of \mathbf{G} is the dihedral group with 8 elements. We denote by $x_r(t)$ ($r \in \Phi$, $t \in \overline{\mathbb{F}_2}$) the Chevalley generators. It is convenient to identify \mathbf{G} with the symplectic group of dimension 4 over the algebraic closure of \mathbb{F}_2 defined by:

$$\text{Sp}_4(\overline{\mathbb{F}_2}) = \{A \in \text{M}_4(\overline{\mathbb{F}_2}) \mid {}^t A J A = J\}, \quad \text{where} \quad J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Representing matrices for the Chevalley generators are, for every $t \in \overline{\mathbb{F}_2}$:

$$x_a(t) = \begin{bmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad x_b(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$x_{a+b}(t) = \begin{bmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad x_{2a+b}(t) = \begin{bmatrix} 1 & 0 & 0 & t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Moreover, we have $x_{-r}(t) = {}^t x_r(t)$. We denote by $\mathbf{H} = \{h(z_1, z_2) \mid z_1, z_2 \in \overline{\mathbb{F}_2}\}$ the subgroup of diagonal matrices of \mathbf{G} and we set $\mathbf{N} = \mathbf{N}_{\mathbf{G}}(\mathbf{H})$. Then for the elements n_a and n_b of \mathbf{N} we have the representing matrices

$$n_a = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad n_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We recall the Chevalley relations of \mathbf{G} ; for every $u, v, z_1, z_2 \in \overline{\mathbb{F}_2}$, we have:

$$\begin{aligned} x_a(u)x_b(v) &= x_b(v)x_a(u)x_{a+b}(uv)x_{2a+b}(u^2v) \\ x_{a+b}(u)x_{2a+b}(v) &= x_{2a+b}(v)x_{a+b}(u) \\ h(z_1, z_2)x_a(u)h(z_1, z_2)^{-1} &= x_a(z_1uz_2^{-1}) \\ h(z_1, z_2)x_b(u)h(z_1, z_2)^{-1} &= x_b(z_2^2u) \\ h(z_1, z_2)x_{a+b}(u)h(z_1, z_2)^{-1} &= x_{a+b}(z_1uz_2) \\ h(z_1, z_2)x_{2a+b}(u)h(z_1, z_2)^{-1} &= x_{2a+b}(z_1^2u) \\ n_a h(z_1, z_2) n_a^{-1} &= h(z_2, z_1) \\ n_b h(z_1, z_2) n_b^{-1} &= h(z_1, z_2^{-1}) \end{aligned} \tag{2}$$

Let n be a positive integer. We define F_{2^n} to be the Frobenius map with parameter 2^n of \mathbf{G} , hence it raises the coefficients of a matrix to their 2^n -th powers. The group \mathbf{G} has a graph endomorphism α described in Proposition 12.3.3 of [3]. It is given on generators by:

$$\begin{aligned} \alpha(x_a(t)) &= x_b(t^2) \\ \alpha(x_b(t)) &= x_a(t) \\ \alpha(x_{a+b}(t)) &= x_{2a+b}(t^2) \\ \alpha(x_{2a+b}(t)) &= x_{a+b}(t) \\ \alpha(h(z_1, z_2)) &= h(z_1 z_2, z_1 z_2^{-1}) \\ \alpha(n_a) &= n_b \\ \alpha(n_b) &= n_a \end{aligned}$$

We fix n a positive integer and we set $\theta = 2^n$ and $q = 2\theta^2$. We define the map:

$$F = F_\theta \circ \alpha.$$

Since $F^2 = F_q$, it follows that F is a generalized Frobenius map of \mathbf{G} . The automorphism of W induced by F has order 2 (that is $\delta = 2$ with the preceding notations). The finite subgroup \mathbf{G}^F is the Suzuki group with parameter q . Using [13], this group is the same as the one studied in [14]. Moreover we have $\mathbf{G}^{F^2} = \mathbf{G}^{F_q} = \text{Sp}_4(\mathbb{F}_q)$. This is a finite "untwisted" group of type B_2 with parameter q . We denote by \mathbf{B} the Borel subgroup of the upper triangular matrices of \mathbf{G} and we set \mathbf{U} to be the unipotent radical of \mathbf{B} . The groups \mathbf{H} , \mathbf{U} and \mathbf{B} are F^2 -stable. We then define $H = \mathbf{H}^{F^2}$, $U = \mathbf{U}^{F^2}$ and $B = \mathbf{B}^{F^2}$. We denote by σ the restriction of F to \mathbf{G}^{F^2} . In the following we set $\mathbf{G}^F = \text{Sz}(q)$ and $\mathbf{G}^{F^2} = G$, and we remark that

$$G^\sigma = \text{Sz}(q).$$

Now the aim is to obtain results about the unipotent characters of $\text{Sz}(q)$ by using Shintani descents between $G \rtimes \langle \sigma \rangle$ and $\text{Sz}(q)$. Before we do this we must compute the irreducible characters of $G \rtimes \langle \sigma \rangle$.

3 The irreducible characters of $B_2(q) \rtimes \langle \sigma \rangle$.

We use the notation of the preceding section. In this section we want to compute the irreducible characters of the extension $\tilde{G} = G \rtimes \langle \sigma \rangle$. This group is an extension by an automorphism of G of order 2. For generalities on character tables of extensions by an automorphism of order 2 we refer to [2]§1. We recall some definitions and general properties. The group G is a normal subgroup of \tilde{G} . Thus a conjugacy class of \tilde{G} is either contained in G or it has no element in G . A class in the first case is called *an inner class* and it is called *an outer class* in the second case. A character ψ of \tilde{G} is called *an outer character* if there exists an outer element (g, σ) such that $\psi(g, \sigma) \neq 0$. We denote by ε the linear character of \tilde{G} with kernel G . Clifford theory shows that the irreducible characters of \tilde{G} can be parameterized by the irreducible characters of G as follows: let $\chi \in \text{Irr}(G)$, then either $\chi^\sigma \neq \chi$ and $\text{Ind}_{\tilde{G}}^{\tilde{G}} \chi \in \text{Irr}(\tilde{G})$, or $\chi^\sigma = \chi$ and χ has two extensions in \tilde{G} which differ up to multiplication by ε . Since the values of extensions on $(1, \sigma)$ are integers, in the case where this value is non-zero, we denote by $\tilde{\chi}$ the extension of χ such that $\tilde{\chi}(1, \sigma) > 0$.

3.1 The outer classes of $B_2(q) \rtimes \langle \sigma \rangle$

The Suzuki group $\text{Sz}(q)$ has three maximal tori that are cyclic groups: $\langle \pi_0 \rangle$, $\langle \pi_1 \rangle$ and $\langle \pi_2 \rangle$ of order $(q-1)$, $(q+2\theta+1)$ and $(q-2\theta+1)$, respectively (see [14]). We denote by E_0 (resp. E_1 and E_2) the set of non-zero classes modulo the equivalence relation \sim on $\mathbb{Z}/(q-1)\mathbb{Z}$ (resp. $\mathbb{Z}/(q+2\theta+1)\mathbb{Z}$ and $\mathbb{Z}/(q-2\theta+1)\mathbb{Z}$) defined by $j \sim i \iff j \equiv \pm i \pmod{(q-1)}$ (resp. $j \equiv \pm i, \pm qi \pmod{(q+2\theta+1)}$ and $j \equiv \pm i, \pm qi \pmod{(q-2\theta+1)}$). We put:

$$E = \{\pi_0^i, \pi_1^j, \pi_2^k \mid i \in E_0, j \in E_1, k \in E_2\}.$$

The conjugacy classes of G are recalled in Table 10 of the appendix. To simplify notation, we denote $x_r(1)$ by x_r , where $r \in \Phi$. We have:

Theorem 3.1 *Let n a non-negative integer. We put $\theta = 2^n$ and $q = 2\theta^2$. Let $G = B_2(q)$ and σ the exceptional automorphism of G that defines $\text{Sz}(q)$. Then the group $\tilde{G} = B_2(q) \rtimes \langle \sigma \rangle$ has $(q+3)$ outer classes. The set*

$$\{(1, \sigma), (x_a, \sigma), (x_{a+b}, \sigma), (x_a x_{a+b}, \sigma), (\pi, \sigma); \pi \in E\}$$

is a system of representatives of the outer classes of \tilde{G} . Moreover, we have:

$$\begin{aligned} |C_{\tilde{G}}(1, \sigma)| &= 2q^2(q-1)(q^2+1), \\ |C_{\tilde{G}}(x_a, \sigma)| &= 4q, \\ |C_{\tilde{G}}(x_{a+b}, \sigma)| &= 2q^2, \\ |C_{\tilde{G}}(x_a x_{a+b}, \sigma)| &= 4q, \\ C_{\tilde{G}}(\pi_0^i, \sigma) &= \langle \pi_0 \rangle \times \langle \sigma \rangle, \\ C_{\tilde{G}}(\pi_1^j, \sigma) &= \langle \pi_1 \rangle \times \langle \sigma \rangle, \\ C_{\tilde{G}}(\pi_2^k, \sigma) &= \langle \pi_2 \rangle \times \langle \sigma \rangle. \end{aligned}$$

Proof — The Suzuki group with parameter q has $(q+3)$ conjugacy classes (see [14]). Using the Shintani correspondence, it follows that \tilde{G} has $(q+3)$ outer classes. Using Table 10, we see that the elements of E are not conjugate in G . Let x be in E . Since $\sigma(x) = x$, we have:

$$C_{\tilde{G}}(x) = C_G(x) \rtimes \langle \sigma \rangle.$$

Moreover, $C_G(x)$ has odd order, hence $|C_{\tilde{G}}(x)|_2 = 2$. Thus $C_{\tilde{G}}(x)$ has a unique class that consists of elements of order 2. We choose $(1, \sigma)$ as a representative of this class. Using the 2-Jordan decomposition of \tilde{G} (see [2] Lemma 3.1), it follows that the elements of $\{(x, \sigma) \mid x \in E\}$ are not conjugate in \tilde{G} . Moreover the group $C_G(x)$ is abelian. Using Lemma 3.2 in [2] and the fact that $C_G(x)^\sigma \cap \langle (1, \sigma) \rangle = \{1\}$, we deduce that:

$$C_{\tilde{G}}(x, \sigma) = C_G(x)^\sigma \times \langle (1, \sigma) \rangle.$$

Furthermore $C_G(x)^\sigma = C_{\text{Sz}(q)}(x)$. In [14] Prop. 16 it is proven that for every $i \in E_0$, $j \in E_1$ and $k \in E_2$, we have $C_{\text{Sz}(q)}(\pi_0^i) = \langle \pi_0 \rangle$, $C_{\text{Sz}(q)}(\pi_1^j) = \langle \pi_1 \rangle$ and $C_{\text{Sz}(q)}(\pi_2^k) = \langle \pi_2 \rangle$. Thus we obtain $(q-1)$ distinct outer classes of \tilde{G} .

Now we prove that $(1, \sigma)$, (x_a, σ) , (x_{a+b}, σ) and $(x_a x_{a+b}, \sigma)$ are not conjugate in \tilde{G} . They are of order 2, 8, 4 and 8 respectively. It then suffices to prove that (x_a, σ) and $(x_a x_{a+b}, \sigma)$ are not conjugate in \tilde{G} . Furthermore using the Bruhat decomposition of G , we can show that two elements in (U, σ) are conjugate in \tilde{G} if and only if they are conjugate by an element of B . Suppose there exists $b = uh \in B$ such that $(b, 1)(x_a, \sigma) = (x_a x_{a+b}, \sigma)(b, 1)$, that is $bx_a = x_a x_{a+b} \sigma(b)$. Let $z_1, z_2 \in \mathbb{F}_q$ such that $h = h(z_1, z_2)$ and we set $z_0 = z_1/z_2$. Then $uhx_a = ux_a(z_0)h$. Moreover:

$$\begin{aligned} ux_a(z_0) &= x_a(t_a)x_b(t_b)x_{a+b}(t_{a+b})x_{2a+b}(t_{2a+b})x_a(z_0) \\ &= x_a(t_a + z_0)x_b(t_b)x_{a+b}(t_{a+b} + z_0t_b)x_{2a+b}(t_{2a+b} + z_0^2t_b), \\ x_a x_{a+b} \sigma(u) &= x_a(1 + t_b^\theta)x_b(t_a^{2\theta})x_{a+b}(1 + t_{2a+b}^\theta + t_a^{2\theta}t_b^\theta) \\ &\quad x_{2a+b}(t_{a+b}^{2\theta} + t_a^{2\theta}t_b^{2\theta}). \end{aligned}$$

By the uniqueness of the decomposition of the elements of B , we deduce that $\sigma(h) = h$ and

$$\begin{cases} t_a + z_0 &= t_b^\theta + 1 \\ t_a^{2\theta} &= t_b \\ z_0t_b + t_{a+b} &= t_a^{2\theta}t_b^\theta + t_{2a+b}^\theta + 1 \\ z_0^2t_b + t_{2a+b} &= t_a^{2\theta}t_b^{2\theta} + t_{a+b}^{2\theta}. \end{cases}$$

Then we obtain $z_0 = 1$ and $z_1 = z_2$. Furthermore since $\sigma(h) = h$ we obtain $z_1 = z_2 = 1$. It follows that $h = 1$. Now we deduce from these relations that $t_b^\theta + t_b + 1 = 0$ and that $t_b^{2\theta} + t_b + 1 = 0$. Thus $t_b^\theta = t_b^{2\theta}$ and t_b^θ is a root of $X^2 + X$, that is $t_b^\theta \in \{0, 1\}$. We then obtain a contradiction in the relation $t_b^\theta + t_b + 1 = 0$. Therefore (x_a, σ) and $(x_a x_{a+b}, \sigma)$ are not conjugate in \tilde{G} .

We now compute the centralizer in \tilde{G} of these elements. First we remark that $C_{\tilde{G}}(\sigma) = \text{Sz}(q) \times \langle \sigma \rangle$. Now let $x \in \{x_a, x_{a+b}, x_{2a+b}\}$. We therefore have $|C_{\tilde{G}}(x, \sigma)| = 2|\{g \in G \mid gx\sigma(g^{-1}) = x\}|$. Using the Bruhat decomposition and a similar calculation as above, we prove that $\{g \in G \mid gx\sigma(g^{-1}) = x\} \subseteq U$.

- The Chevalley relations give

$$\begin{cases} t_a &= t_b^\theta \\ t_{2a+b}^\theta + t_a^{2\theta} t_b^\theta &= t_b + t_{a+b} \\ t_{a+b}^{2\theta} + t_a^{2\theta} t_b^{2\theta} &= t_b + t_{2a+b} \end{cases}.$$

It follows that $t_b \in \{0, 1\}$. In both cases the number of solutions is

$$|\{(t_{2a+b}, t_{a+b}) \mid t_{2a+b}^\theta = t_{a+b}\}| = q.$$

Finally we have: $|C_{\tilde{G}}(x_a, \sigma)| = |C_{\tilde{G}}(x_a x_{a+b}, \sigma)| = 4q$.

- We have

$$\{g \in G \mid gx_{a+b}\sigma(g^{-1}) = x_{a+b}\} = \{u \in U \mid \sigma(u) = u\} = U \cap \text{Sz}(q).$$

Furthermore $|U \cap \text{Sz}(q)| = q^2$. We thus deduce that $|C_{\tilde{G}}(x_{a+b}, \sigma)| = 2q^2$.

□

3.2 The σ -stable characters of $B_2(q)$

In Table 11 of the appendix we recall the values of the irreducible characters of $B_2(q)$ that we need in this work. We have:

Proposition 3.1 *The group $B_2(q)$ has $(q+3)$ σ -stable irreducible characters:*

- The σ -stable unipotent characters 1_G , θ_1 , θ_4 and θ_5 of degree respectively 1 , $\frac{1}{2}q(q+1)^2$, q^4 and $\frac{1}{2}q(q-1)^2$.
- The $\frac{1}{2}(q-2)$ characters $\chi_1(i, (2\theta-1)i)$, $i \in E_0$ of degree $(q+1)^2(q^2+1)$. We denote these characters by $\chi_{\pi_0}(i)$.
- The $\frac{1}{4}(q+2\theta)$ characters $\chi_5((q-2\theta+1)j)$, $j \in E_1$ of degree $(q^2-1)^2$, denoted by $\chi_{\pi_1}(j)$.
- The $\frac{1}{4}(q-2\theta)$ characters $\chi_5((q+2\theta+1)k)$, $k \in E_2$ of degree $(q^2-1)^2$, denoted by $\chi_{\pi_2}(k)$.

3.3 Irreducible characters obtained by induction from $B \rtimes \langle \sigma \rangle$

The group B is σ -stable, thus $\tilde{B} = B \rtimes \langle \sigma \rangle \subseteq \tilde{G}$. We now induce some characters of \tilde{B} to \tilde{G} which permit to obtain the outer values of $\tilde{\chi}_{\pi_0}(i)$ ($i \in E_0$) and of $\tilde{\theta}_4$. Let γ_0 the primitive $(q-1)$ -th root of unity given in Table 11. We define the primitive $(q-1)$ -th root of unity $\varepsilon_0 = \gamma_0^{(4-4\theta)}$ and we set $\varepsilon_0^i(\pi_0^l) = \varepsilon_0^{il} + \varepsilon_0^{-il}$.

Proposition 3.2 *We have:*

	$(1, \sigma)$	(x_a, σ)	(x_{a+b}, σ)	$(x_a x_{a+b}, \sigma)$	(π_0^l, σ)	(π_1^m, σ)	(π_2^r, σ)
$\tilde{\chi}_{\pi_0}(i)$	$q^2 + 1$	1	1	1	$\varepsilon_0^i(\pi_0^l)$	0	0
$\tilde{\theta}_4$	q^2	0	0	0	1	-1	-1

Proof — We have $U \triangleleft \tilde{B}$ and we denote by $\pi_U : \tilde{B} \rightarrow \tilde{B}/U$ the canonical map. Let $\phi \in \text{Irr}(\tilde{B}/U)$; then $\phi \circ \pi_U$ is an irreducible character of \tilde{B} . We have $\tilde{B}/U \simeq H \rtimes \langle \sigma \rangle$. We now construct outer characters of $H \rtimes \langle \sigma \rangle$. Since $H \simeq \mathbb{F}_q^\times \times \mathbb{F}_q^\times$, it follows that the irreducible characters of H are:

$$\forall 1 \leq k, l \leq q-1 \quad \phi_{k,l}(\gamma^i, \gamma^j) = \gamma_0^{ik+jl}.$$

The σ -stable irreducible characters of H are $\phi_{i,(2\theta-1)i}$ ($i \in \{1, \dots, q-1\}$). Using [2] Lemma 3.5, we construct $(q-2)$ linear characters of $H \rtimes \langle \sigma \rangle$ defined by $\varphi_i(h, x) = \phi_{i,(2\theta-1)i}(h)$, where $x \in \{1, \sigma\}$. We write $\phi_{i,\tilde{B}} = \varphi_i \circ \pi_U$; we thus obtain $(q-2)$ linear characters of \tilde{B} . Moreover \tilde{B} has $(q+2)$ outer classes, which are (π_0^l, σ) ($l \in \{1, (q-2)\}$) and $(1, \sigma)$, (x_a, σ) , (x_{a+b}, σ) and $(x_a x_{a+b}, \sigma)$. It is then easy to compute the values of $\phi_{i,\tilde{B}}$. Moreover using the Mackey formula, it follows that $\text{Ind}_{\tilde{B}}^{\tilde{G}} \phi_{i,\tilde{B}}$ is an irreducible extension of $\chi_{\pi_0}(i)$. To obtain the outer values of these characters, we will induce $\phi_{i,\tilde{B}}$ from \tilde{B} to \tilde{G} . To this end, we give the corresponding induction formula. Except for $(1, \sigma)$, the centralizers of the outer elements of \tilde{B} are the same as their centralizers in \tilde{G} . We have $C_{\tilde{B}}(1, \sigma) = (\text{Sz}(q) \cap B, \sigma) = B^\sigma \times \langle \sigma \rangle$ of order $2q^2(q-1)$ and

$$\text{Cl}_{\tilde{G}}(\pi_0, \sigma) \cap \tilde{B} = \text{Cl}_{\tilde{B}}(\pi_0, \sigma) \cup \text{Cl}_{\tilde{B}}(\pi_0^{-1}, \sigma).$$

This permits to obtain the induction formula from \tilde{B} to \tilde{G} given in the following table:

	$(1, \sigma)$	(x_a, σ)	(x_{a+b}, σ)	$(x_a x_{a+b}, \sigma)$	(π_0, σ)
$\text{Ind}_{\tilde{B}}^{\tilde{G}} \phi$	$(q^2 + 1)\phi(1, \sigma)$	$\phi(x_a, \sigma)$	$\phi(x_{a+b}, \sigma)$	$\phi(x_a x_{a+b}, \sigma)$	$\phi(\pi_0, \sigma) + \phi(\pi_0^{-1}, \sigma)$

Using this formula, we compute the values of $\tilde{\chi}_{\pi_0}(i)$.

We have:

$$\begin{aligned} \langle \text{Ind}_{\tilde{B}}^{\tilde{G}} 1_B, 1_G \rangle_G &= 1 & \langle \text{Ind}_{\tilde{B}}^{\tilde{G}} 1_B, \theta_1 \rangle_G &= 2 \\ \langle \text{Ind}_{\tilde{B}}^{\tilde{G}} 1_B, \theta_4 \rangle_G &= 1 & \langle \text{Ind}_{\tilde{B}}^{\tilde{G}} 1_B, \theta_5 \rangle_G &= 0 \\ \langle \text{Ind}_{\tilde{B}}^{\tilde{G}} 1_B, \chi \rangle_G &= 0 & \forall \chi = \chi^\sigma \text{ such that } \chi \neq 1, \theta_4, \theta_1, \theta_5. \end{aligned}$$

	$\text{Ind}_{\text{Sz}(q)}^G \phi$		$\text{Ind}_{\widetilde{\text{Sz}}(q)}^{\widetilde{G}} \phi$
A_1	$q^2(q+1)(q^2-1)\phi(1)$	(π_0, σ)	$\phi(\pi_0, \sigma)$
A_{32}	$q^2\phi(\sigma_0)$	$(1, \sigma)$	$\phi(1, \sigma) + (q-1)(q^2+1)\phi(\sigma_0, \sigma)$
A_{42}	$q(\phi(\rho_0) + \phi(\rho_0^{-1}))$	(x_a, σ)	0
π_0	$(q-1)\phi(\pi_0)$	(x_{a+b}, σ)	$\frac{q}{2}(\phi(\rho_0, \sigma) + \phi(\rho_0^{-1}, \sigma))$
π_1	$(q-r+1)\phi(\pi_1)$	$(x_a x_{a+b}, \sigma)$	0
π_2	$(q+r+1)\phi(\pi_2)$	(π_1, σ)	$\phi(\pi_1, \sigma)$
		(π_2, σ)	$\phi(\pi_2, \sigma)$

Table 1: Induction formula from $\widetilde{\text{Sz}}(q)$ to \widetilde{G} .

We have $\text{Res}_{\widetilde{G}}^{\widetilde{G}}(\text{Ind}_{\widetilde{B}}^{\widetilde{G}} 1_{\widetilde{B}}) = \text{Ind}_B^G 1_B$ and write ξ for the constituent of $\text{Ind}_{\widetilde{B}}^{\widetilde{G}} 1_{\widetilde{B}}$ which is an extension of θ_1 . Since $\langle \text{Ind}_B^G 1_B, \theta_1 \rangle_G = 2$, it follows that ξ occurs in $\text{Ind}_{\widetilde{B}}^{\widetilde{G}} 1_{\widetilde{B}}$ with multiplicity 1 or 2. If this multiplicity is 2, then

$$\langle \text{Ind}_{\widetilde{B}}^{\widetilde{G}} 1_{\widetilde{B}}, \text{Ind}_{\widetilde{B}}^{\widetilde{G}} 1_{\widetilde{B}} \rangle_{\widetilde{G}} \geq 6.$$

But a direct calculation gives $\langle \text{Ind}_{\widetilde{B}}^{\widetilde{G}} 1_{\widetilde{B}}, \text{Ind}_{\widetilde{B}}^{\widetilde{G}} 1_{\widetilde{B}} \rangle_{\widetilde{G}} = 5$. Thus it follows that $\xi + \varepsilon\xi$ is a constituent of $\text{Ind}_{\widetilde{B}}^{\widetilde{G}} 1_{\widetilde{B}}$. Decomposing the character $\text{Ind}_{\widetilde{B}}^{\widetilde{G}}(1_{\widetilde{B}}) - 1_{\widetilde{G}} - \xi - \varepsilon\xi$, we find an irreducible extension of θ_4 . We use the preceding induction formula to compute its values.

□

3.4 Irreducible characters obtained by induction from $\text{Sz}(q) \times \langle \sigma \rangle$

Let $\rho_0 = x_a x_{a+b} \in \text{Sz}(q)$ and $\sigma_0 = x_{a+b} x_{2a+b} \in \text{Sz}(q)$. Let τ_0 be the complex primitive root of order $q^2 + 1$ which appears in the character table of G . We recall that $\{1, \sigma_0, \rho_0, \rho_0^{-1}, \pi_0^i; i \in E_0, \pi_1^j; j \in E_1, \pi_2^k; k \in E_2\}$ is a system of representatives of the classes of $\text{Sz}(q)$ (see [14]). We put $\varepsilon_1 = \tau_0^{(q-2\theta+1)^2}$ and $\varepsilon_2 = \tau_0^{(q+2\theta+1)^2}$. The character Table of $\text{Sz}(q)$ is computed in [14] and reprinted in the appendix for the convenience of the reader. Since $\text{Sz}(q)$ is the subgroup of fixed points under σ , it follows that $\widetilde{\text{Sz}}(q) = \text{Sz}(q) \times \langle \sigma \rangle \subseteq \widetilde{G}$. Thus the classes and the character table of $\widetilde{\text{Sz}}(q)$ are directly obtained using the classes and the character table of $\text{Sz}(q)$. We give the induction formula from $\widetilde{\text{Sz}}(q)$ to \widetilde{G} in Table 1. Let $\widetilde{\phi} \in \text{Irr}(\widetilde{\text{Sz}}(q))$; to simplify we denote by the same symbol its induced character of \widetilde{G} .

	A_1	A_{32}	A_{42}
$\text{Ind}_{\text{Sz}(q)}^G 1_{\text{Sz}(q)}$	$q^2(q+1)(q^2-1)$	q^2	$2q$
$\text{Ind}_{\text{Sz}(q)}^G \text{St}$	$q^4(q+1)(q^2-1)$	0	0
$\text{Ind}_{\text{Sz}(q)}^G X_i$	$q^2(q^2+1)(q+1)(q^2-1)$	q^2	$2q$
$\text{Ind}_{\text{Sz}(q)}^G Y_j$	$q^2(q-\theta+1)(q^2-1)^2$	$q^2(\theta-1)$	$-2q$
$\text{Ind}_{\text{Sz}(q)}^G Z_k$	$q^2(q+\theta+1)(q^2-1)^2$	$-q^2(\theta+1)$	$-2q$
$\text{Ind}_{\text{Sz}(q)}^G \mathcal{W}$	$\frac{1}{2}q^2\theta(q^2-1)^2$	$-q^2\theta$	0

	π_0	π_1	π_2
$\text{Ind}_{\text{Sz}(q)}^G 1$	$q-1$	$q-\theta+1$	$q+\theta+1$
$\text{Ind}_{\text{Sz}(q)}^G \text{St}$	$q-1$	$-(q-\theta+1)$	$-(q+\theta+1)$
$\text{Ind}_{\text{Sz}(q)}^G X_i$	$(q-1)(\varepsilon_0^i \pi_0)$	0	0
$\text{Ind}_{\text{Sz}(q)}^G Y_j$	0	$-(q-\theta+1)\varepsilon_1^j(\pi_1)$	0
$\text{Ind}_{\text{Sz}(q)}^G Z_k$	0	0	$-(q+\theta+1)\varepsilon_2^k(\pi_2)$
$\text{Ind}_{\text{Sz}(q)}^G \mathcal{W}$	0	$q-\theta+1$	$-(q+\theta+1)$

Table 2: Values of the induced character of $\text{Sz}(q)$.

Proposition 3.3 *We set $\varepsilon_0^l(\pi_0^i) = \varepsilon_0^{il} + \varepsilon_0^{-il}$, $\varepsilon_1^l(\pi_1^j) = \varepsilon_1^{jl} + \varepsilon_1^{-jl} + \varepsilon_1^{jlq} + \varepsilon_1^{-jlq}$ and $\varepsilon_2^l(\pi_2^k) = \varepsilon_2^{kl} + \varepsilon_2^{-kl} + \varepsilon_2^{klq} + \varepsilon_2^{-klq}$. In Tables 2 and 3, we give the values of the induced characters from $\widetilde{\text{Sz}}(q)$ to \widetilde{G} .*

Let ψ be a generalized character of \widetilde{G} . We recall that we can associate to ψ its σ -reduction $\rho(\psi)$, which is a character of \widetilde{G} such that $\rho(\psi)(g, \sigma) = \psi(g, \sigma)$ (for every $g \in G$) and the irreducible constituents of $\rho(\psi)$ are outer characters of \widetilde{G} . We refer to [2] §3.2.1 for details. We now define:

$$\begin{aligned}
X_0 &= \rho \left(\widetilde{1}_{\text{Sz}(q)} - 1_{\widetilde{G}} - \widetilde{\theta}_4 - \sum_{i \in E_0} \widetilde{\chi}_{\pi_0}(i) \right), \\
W_0 &= \rho \left(\varepsilon \widetilde{W} - \theta(\widetilde{1}_{\text{Sz}(q)} - 1_{\widetilde{G}}) \right).
\end{aligned}$$

Proposition 3.4 *For every $k \in E_1$ (resp. $k \in E_2$), there exists an extension ψ_k*

	$(1, \sigma)$	(x_{a+b}, σ)	(π_0, σ)	(π_1, σ)	(π_2, σ)
$\widetilde{1}_{\text{Sz}(q)}$	$q(q^2 - q + 1)$	q	1	1	1
$\widetilde{\text{St}}$	q^2	0	1	-1	-1
\widetilde{X}_i	$q(q^2 + 1)$	q	$\varepsilon_0^i(\pi_0)$	0	0
\widetilde{Y}_j	$q - 1)(q + q^2(\theta - 1))$	$-q$	0	$-\varepsilon_1^j(\pi_1)$	0
\widetilde{Z}_k	$(q - 1)(q - q^2(\theta + 1))$	$-q$	0	0	$-\varepsilon_2^k(\pi_2)$
$\widetilde{\mathcal{W}}$	$-\frac{1}{2}q^2\theta(q - 1)$	0	0	1	-1

Table 3: Outer values of the induced characters of $\widetilde{\text{Sz}}(q)$

(resp. ψ'_k) of $\chi_{\pi_1}(k)$ (resp. $\chi_{\pi_2}(k)$) such that:

$$X_0 = \sum_{k \in E_1} \psi_k + \sum_{k \in E_2} \psi'_k,$$

$$W_0 = \sum_{k \in E_1} \psi_k - \sum_{k \in E_2} \psi'_k.$$

We give the values of X_0 and W_0 in the following table:

	$(1, \sigma)$	(x_a, σ)	(x_{a+b}, σ)	$(x_a x_{a+b}, \sigma)$	(π_0, σ)	(π_1, σ)	(π_2, σ)
X_0	$q(q - 1)^2/2$	$-q/2$	$q/2$	$-q/2$	0	1	1
W_0	$-\theta(q - 1)$	θ	$-\theta(q - 1)$	θ	0	-1	1

Proof — By computing the scalar products of $\widetilde{1}_{\text{Sz}(q)}$ with the irreducible characters obtained in Prop. 3.2, we deduce that $\widetilde{1}_{\text{Sz}(q)} - 1_{\widetilde{G}} - \widetilde{\theta}_4 - \sum_{i \in E_0} \widetilde{\chi}_{\pi_0}(i)$ is a character. Furthermore, we have

$$\begin{aligned} \langle \text{Ind}_{\text{Sz}(q)}^G(1_{\text{Sz}(q)}), \theta_1 \rangle_G &= \langle \text{Ind}_{\text{Sz}(q)}^G(1_{\text{Sz}(q)}), \theta_5 \rangle_G = 0 \\ \langle \text{Ind}_{\text{Sz}(q)}^G(1_{\text{Sz}(q)}), \chi_{\pi_1}(k) \rangle_G &= \langle \text{Ind}_{\text{Sz}(q)}^G(1_{\text{Sz}(q)}), \chi_{\pi_2}(k) \rangle_G = 1. \end{aligned}$$

This proves that X_0 is exactly the summand of one extension of $\chi_{\pi_1}(k)$ (denoted by ψ_k) and one extension of $\chi_{\pi_2}(k)$ (denoted by ψ'_k). Now we compute the scalar products of $\widetilde{\mathcal{W}}$ with the known irreducible characters of \widetilde{G} . Thus $\widetilde{\mathcal{W}} - \theta(\widetilde{\theta}_4\varepsilon + \sum \widetilde{\chi}_{\pi_0}(i)\varepsilon)$ is a character of \widetilde{G} and we denote by Θ its σ -reduction. We compute that $\langle \text{Res}_{\widetilde{G}}^{\widetilde{G}} \widetilde{\mathcal{W}}, \theta_1 \rangle_G = \langle \text{Res}_{\widetilde{G}}^{\widetilde{G}} \widetilde{\mathcal{W}}, \theta_5 \rangle_G = 0$, $\langle \text{Res}_{\widetilde{G}}^{\widetilde{G}} \widetilde{\mathcal{W}}, \chi_{\pi_1}(k) \rangle_G = \theta - 1$ and $\langle \text{Res}_{\widetilde{G}}^{\widetilde{G}} \widetilde{\mathcal{W}}, \chi_{\pi_2}(k) \rangle_G = \theta + 1$. Since X_0 and $\widetilde{\mathcal{W}}$ have no common constituents (because $\langle X_0, \widetilde{\mathcal{W}} \rangle_{\widetilde{G}} = 0$) we deduce that $\Theta = (\theta - 1) \sum_{k \in E_1} \psi_k \varepsilon +$

$(\theta + 1) \sum_{k \in E_2} \psi'_k \varepsilon$. We remark that $W_0 = \Theta \varepsilon - \theta X_0$ and deduce that

$$W_0 = \sum_{k \in E_2} \psi'_k - \sum_{k \in E_1} \psi_k.$$

□

For every $j \in E_1$ and $k \in E_2$, we put $\varphi_j = \rho(\tilde{Y}_j - \tilde{Y}_1)$ and $\vartheta_k = \rho(\tilde{Z}_k - \tilde{Z}_1)$.

Proposition 3.5 *Using the preceding notation we have:*

$$\varepsilon \tilde{\chi}_{\pi_1}(1) = \frac{4}{q+2\theta} \left(\sum_{j \in E_1} \varphi_j + \frac{1}{2}(X_0 - W_0) \right),$$

$$\varepsilon \tilde{\chi}_{\pi_2}(1) = \frac{4}{q-2\theta} \left(\sum_{k \in E_2} \vartheta_k + \frac{1}{2}(X_0 + W_0) \right).$$

Consequently we deduce that:

	$(1, \sigma)$	(x_a, σ)	(x_{a+b}, σ)	$(x_a x_{a+b}, \sigma)$	(π_1, σ)	(π_2, σ)
$\tilde{\chi}_{\pi_1}(j)$	$(q-1)(q-2\theta+1)$	-1	$2\theta-1$	-1	$-\varepsilon_1^j(\pi_1)$	0
$\tilde{\chi}_{\pi_2}(k)$	$(q-1)(q+2\theta+1)$	-1	$-2\theta-1$	-1	0	$-\varepsilon_2^k(\pi_2)$

Proof — We have $\langle \text{Res}_{\tilde{G}}^{\tilde{G}}(\varphi_j), \chi_{\pi_1}(j) \rangle_G = -1$ and $\langle \text{Res}_{\tilde{G}}^{\tilde{G}}(\varphi_j), \chi_{\pi_1}(1) \rangle_G = 1$. Moreover for every other σ -stable character χ of G we have $\langle \text{Res}_{\tilde{G}}^{\tilde{G}}(\varphi_j), \chi \rangle_G = 0$. This yields $\text{Res}_{\tilde{G}}^{\tilde{G}}(\varphi_j) = \chi_{\pi_1}(1) - \chi_{\pi_1}(j)$. We similarly prove that $\text{Res}_{\tilde{G}}^{\tilde{G}}(\vartheta_k) = \chi_{\pi_2}(1) - \chi_{\pi_2}(k)$. We denote by θ_j (resp. $\theta_{1,j}$) the extension of $\chi_{\pi_1}(j)$ (resp. $\chi_{\pi_1}(1)$), which is an irreducible component of φ_j . In particular we have $\varphi_j = \theta_{1,j} - \theta_j$. First we prove that $\theta_{1,j}$ is independent of j . Indeed, we immediately compute that $\langle \varphi_j - \varphi_2, \varphi_j - \varphi_2 \rangle_{\tilde{G}} = 2$ (where $j \geq 3$). Furthermore we have $\varphi_j - \varphi_2 = \theta_2 - \theta_j + \theta_{1,j} - \theta_{1,2}$. Moreover since $\text{Res}_{\tilde{G}}^{\tilde{G}}(\varphi_j - \varphi_2) = \chi_{\pi_1}(2) - \chi_{\pi_1}(j)$, the characters θ_2 and θ_j are constituents of $\varphi_j - \varphi_2$. Since $\varphi_j - \varphi_2$ has two constituents, it follows that $\theta_{1,j} - \theta_{1,2} = 0$, i.e.,

$$\forall j \geq 2 \quad \theta_{1,j} = \theta_{1,2}.$$

We denote by θ_1 this common constituent. We compute that $\langle \varphi_i, X_0 \rangle_{\tilde{G}} = 0$. Also, if $\theta_1 = \psi_1$ then for every $j \geq 2$, we have $\theta_j = \psi_j$. Indeed, if this is not the case we have $\langle \varphi_i, X_0 \rangle_{\tilde{G}} = 1 \neq 0$. With a similar argument we prove that if $\theta_1 = \varepsilon \psi_1$ then for every $j \geq 2$, we obtain $\theta_j = \varepsilon \psi_j$. In summary we have:

$$\begin{cases} \theta_j = \psi_j & \forall j \in E_1 \\ \text{or} \\ \theta_j = \psi_j \varepsilon & \forall j \in E_1 \end{cases}.$$

Moreover,

$$\frac{1}{4}(q + 2\theta)\theta_1 = \frac{1}{4}(q + 2\theta)\theta_1 - \sum_{j \in E_1} \theta_j + \sum_{i \in E_1} \theta_j = \sum_{j \neq 1} \varphi_j + \sum_{j \in E_1} \theta_j.$$

Thus

$$\theta_1 = \frac{4}{q + 2\theta} \left(\sum_{j \neq 1} \varphi_j + \sum_{j \in E_1} \theta_j \right).$$

We immediately deduce from Proposition 3.4 that,

$$\sum_{j \in E_1} \psi_j = \frac{1}{2}(X_0 - W_0).$$

Hence we have,

$$\begin{cases} \sum_{j \in E_1} \theta_j &= \frac{1}{2}(X_0 - W_0) \\ &\text{or} \\ \sum_{j \in E_1} \theta_j &= \frac{1}{2}(X_0 - W_0)\varepsilon \end{cases}.$$

We set

$$\begin{cases} f_1 &= \frac{4}{q + 2\theta} \left(\sum_{j \neq 1} \varphi_j + \frac{1}{2}(X_0 - W_0) \right), \\ f_2 &= \frac{4}{q + 2\theta} \left(\sum_{j \neq 1} \varphi_j + \frac{1}{2}(X_0 - W_0)\varepsilon \right). \end{cases}$$

Either f_1 or f_2 is an irreducible character of \tilde{G} . But $f_1(\pi_1, \sigma)$ is not an algebraic integer. Thus f_1 is not a character. It follows that $f_2 = \varepsilon\psi_1$. We similarly prove that

$$\varepsilon\psi'_1 = \frac{4}{q - 2\theta} \left(\sum_{k \neq 1} \vartheta_k + \frac{1}{2}(X_0 + W_0)\varepsilon \right).$$

Since we have $\psi_j = \varepsilon\phi_j$, we immediately deduce the values of ψ_j using the relation $\psi_j = \psi_1 - \varepsilon\varphi_j$. Similarly we compute the values of ψ'_k using the relation $\psi'_k = \psi'_1 - \varepsilon\vartheta_k$. □

The induced characters of $\tilde{\text{Sz}}(q)$ are not sufficient to obtain the outer values of the extensions of $\tilde{\theta}_1$ and $\tilde{\theta}_5$. However decomposing \tilde{Y}_i we obtain:

Lemma 3.1 *The outer values of $\tilde{\theta}_1 + \tilde{\theta}_5$ are:*

	$(1, \sigma)$	(x_a, σ)	(x_{a+b}, σ)	$(x_a x_{a+b}, \sigma)$	(π_0, σ)	(π_1, σ)	(π_2, σ)
$\tilde{\theta}_1 + \tilde{\theta}_5$	$2\theta(q - 1)$	0	-2θ	0	0	2	-2

3.5 The outer values of $\tilde{\theta}_1$ and $\tilde{\theta}_5$

Let $U_0 = \langle x_a \rangle \langle x_b \rangle X_{a+b} X_{2a+b} \subseteq U$. Since U_0 is σ -stable, we have $\tilde{U}_0 = U_0 \rtimes \langle \sigma \rangle \subseteq \tilde{B}$. We recall that the irreducible characters of B are given on p. 87 of [6]. We use the same notation. Let $\theta_3(1)$ be the irreducible character of B of degree $\frac{1}{2}q(q-1)^2$. Using the character tables of G and B we deduce that $\text{Res}_B^G \theta_5 = \theta_3(1)$. Thus $\text{Res}_B^{\tilde{G}} \tilde{\theta}_5 = \tilde{\theta}_3(1)$. We now construct $\tilde{\theta}_3(1)$ using \tilde{U}_0 . We set $\lambda : \mathbb{F}_q \longrightarrow \{-1, 1\}$, such that $\lambda(x) = 1$ if $X^2 + X + x$ has a root in \mathbb{F}_q and $\lambda(x) = -1$ otherwise. Let $k, l \in \{0, 1\}$ and define the linear character $\lambda(k, l)$ of U_0 on the generators by $\lambda(k, l)(x_a) = (-1)^k$, $\lambda(k, l)(x_b) = (-1)^l$ and $\lambda(k, l)(x_{a+b}(u)x_{2a+b}(v)) = \lambda(u + v)$.

Lemma 3.2 *The characters $\lambda(0, 0)$ and $\lambda(1, 1)$ are σ -stable.*

Proof — Since $\lambda(v) = 1$ if and only if $\lambda(v^\theta) = 1$ and since $\lambda(v) = -1$ if and only if $\lambda(v^\theta) = -1$, it follows that

$$\lambda(k, l)^\sigma(x_{a+b}(u)x_{2a+b}(v)) = \lambda(k, l)(x_{a+b}(u)x_{2a+b}(v)).$$

We remark that if $(k, l) \in \{(0, 0), (1, 1)\}$, then the map has the same values on x_a and x_b . Thus $\lambda(0, 0)$ and $\lambda(1, 1)$ are σ -stable. \square

Using [2] Lemma 3.5 the characters $\lambda(0, 0)$ and $\lambda(1, 1)$ can be extend to \tilde{U}_0 and we thus obtain $\tilde{\lambda}(0, 0)$ and $\tilde{\lambda}(1, 1)$.

Proposition 3.6 *The conjugacy classes of U_0 (resp. of \tilde{U}_0) are given in Table 4 (resp. in Table 5). In tables 6 and 7, we give the induction formula of \tilde{U}_0 to \tilde{B} .*

Proof — We remark that $\text{Ker } \lambda = \{t + t^\theta \mid t \in \mathbb{F}_q\}$. Thus using [6] and the Chevalley relations, we obtain the fusion of classes. The result then follows. \square

Lemma 3.3 *The values of $\text{Ind}_{\tilde{U}_0}^{\tilde{B}} \tilde{\lambda}(1, 1)$ are given in Table 8.*

Proof — It suffices to use Proposition 3.6 and the relation $\sum_{t \in \mathbb{F}_q} \lambda(t) = 0$. \square

Lemma 3.4 *There exists an extension ξ of $\theta_3(1)$ such that*

$$\text{Ind}_{\tilde{U}_0}^{\tilde{B}} \tilde{\lambda}(1, 1) = \frac{1}{4}(q + 2\theta)\xi + \frac{1}{4}(q - 2\theta)\xi\varepsilon.$$

Proof — We set $\chi = \text{Ind}_{\tilde{U}_0}^{\tilde{B}} \tilde{\lambda}(1, 1)$. Using the character table of B (see [6] p.87) we prove that $\langle \text{Res}_B^{\tilde{B}} \chi, \theta_3(1) \rangle_B = \frac{1}{2}q$. Thus there exist an extension ξ of $\theta_3(1)$,

	Representatives	Number	$ C_{U_0}(x) $
	1	1	$4q^2$
	x_a	1	$2q^2$
	x_b	1	$2q^2$
	$x_{a+b}(u)$	$q-1$	$4q^2$
	$x_{2a+b}(v)$	$q-1$	$4q^2$
	$x_a x_b$	1	$2q^2$
	$x_{a+b}(u)x_{2a+b}(v)$	$(q-1)^2$	$4q^2$
$u \neq 0$	$x_a x_{a+b}(u)$	$q-1$	$2q^2$
$u \neq 0$	$x_b x_{a+b}(u)$	$q-1$	$2q^2$
$v \neq 0, 1$	$x_a x_{2a+b}(v)$	$q-2$	$2q^2$
$v \neq 0, 1$	$x_b x_{2a+b}(v)$	$q-2$	$2q^2$
$u \neq 0$	$x_a x_b x_{a+b}(u)$	$q-1$	$2q^2$
$v \neq 0, 1$	$x_a x_b x_{a+b}(v)$	$q-2$	$2q^2$
$u, v \neq 0, 1$	$x_a x_{a+b}(u) b x_{2a+b}(v)$	$\frac{1}{2}(q-2)^2$	$2q^2$
$u, v \neq 0, 1$	$x_b x_{a+b}(u) b x_{2a+b}(v)$	$\frac{1}{2}(q-2)^2$	$2q^2$
$u, v \neq 0, 1$	$x_a x_b x_{a+b}(u) b x_{2a+b}(v)$	$\frac{1}{2}(q-2)^2$	$2q^2$

Table 4: Conjugacy classes of U_0 .

	Representatives	Number	$ C_{\tilde{U}_0}(x) $
	$(1, \sigma)$	1	$4q$
	$(x_a x_{a+b}(u), \sigma)$	q	$4q$
$u \neq 0$	$(x_{a+b}(u), \sigma)$	$q-1$	$4q$

Table 5: Outer classes of \tilde{U}_0 .

integers n and n_ε where $n \geq n_\varepsilon$, such that $n + n_\varepsilon = \frac{1}{2}q$ and $\chi = n\xi + n_\varepsilon\xi\varepsilon$. We have $\chi\varepsilon = n_\varepsilon\xi + n\xi\varepsilon$. We deduce that $\chi - \chi\varepsilon = (n - n_\varepsilon)(\xi - \xi\varepsilon)$. It follows that $\langle \chi - \chi\varepsilon, \chi - \chi\varepsilon \rangle_{\tilde{B}} = 2(n - n_\varepsilon)^2$. Furthermore, since $\langle \chi - \chi\varepsilon, \chi - \chi\varepsilon \rangle_{\tilde{B}} = q$ we get $2(n - n_\varepsilon)^2 = 2\theta^2$ and hence $(n - n_\varepsilon)^2 = \theta^2$. This yields $n - n_\varepsilon = \theta$. Solving the system

$$\begin{cases} n + n_\varepsilon &= \frac{1}{2}q \\ n - n_\varepsilon &= \theta \end{cases}$$

we find $n = \frac{1}{4}(q + 2\theta)$ and $n_\varepsilon = \frac{1}{4}(q - 2\theta)$.

□

Lemma 3.5 *The outer values of $\tilde{\theta}_3(1)$ are:*

	$(1, \sigma)$	(x_a, σ)	(x_{a+b}, σ)	$(x_a x_{a+b}, \sigma)$	(π_0, σ)
$\tilde{\theta}_3(1)$	$\theta(q-1)$	$-\theta$	$-\theta$	θ	0

Proof — Using Lemma 3.4, we compute the values of ξ . For example to compute $\xi(1, \sigma)$ we set $\xi(1, \sigma) = \alpha$. Then we have:

$$\frac{1}{4}(q + 2\theta)\alpha - \frac{1}{4}(q - 2\theta)\alpha = \frac{1}{2}q(q - 1),$$

and we deduce $\alpha = \theta(q - 1)$. Moreover, since $\xi(1, \sigma) > 0$, it follows that $\tilde{\theta}_3(1) = \xi$. □

We now set $\phi = \text{Res}_{\tilde{B}}^{\tilde{G}}(\tilde{\theta}_1 + \tilde{\theta}_5)$. We have $\langle \phi, \tilde{\theta}_3(1) \rangle_{\tilde{B}} = 1$ and $\langle \phi, \varepsilon\tilde{\theta}_3(1) \rangle_{\tilde{B}} = 0$. Moreover $\text{Res}_{\tilde{B}}^{\tilde{G}}\tilde{\theta}_5$ is an extension of $\theta_3(1)$. Thus $\text{Res}_{\tilde{B}}^{\tilde{G}}\tilde{\theta}_5 = \tilde{\theta}_3(1)$ and $\text{Res}_{\tilde{B}}^{\tilde{G}}\tilde{\theta}_1 = \phi - \tilde{\theta}_3(1)$. We then obtain the values of $\tilde{\theta}_5$ and $\tilde{\theta}_1$ on (π_0, σ) , $(1, \sigma)$, (x_a, σ) , (x_{a+b}, σ) and $(x_a x_{a+b}, \sigma)$. This leaves us with the values of $\tilde{\theta}_1$ and $\tilde{\theta}_5$ on (π_1, σ) and (π_2, σ) which still need to be computed.

Lemma 3.6 *We have $\tilde{\theta}_1(\pi_1, \sigma) = \tilde{\theta}_5(\pi_1, \sigma) = 1$ and $\tilde{\theta}_1(\pi_2, \sigma) = \tilde{\theta}_5(\pi_2, \sigma) = -1$.*

Proof — Setting $\alpha = \tilde{\theta}_1(\pi_1, \sigma)$ and $\beta = \tilde{\theta}_5(\pi_1, \sigma)$ we have $\alpha + \beta = 2$. The orthogonality relations of the rows give $|\alpha|^2 + |\beta|^2 = 2$. Since (π_1, σ) and $(\pi_1, \sigma)^{-1}$ are conjugate, it follows that α and β are real numbers. Substituting β by $2 - \alpha$ we see that α is a root of $X^2 - 2X + 1$ and therefore $\alpha = 1$. It follows that $\beta = 1$. Similarly we have $\tilde{\theta}_1(\pi_2, \sigma) = \tilde{\theta}_5(\pi_2, \sigma) = -1$. □

In particular through Lemmas 3.5 and 3.6 and Propositions 3.2 and 3.5 we obtain the character table of \tilde{G} . Thus Theorem 1.1 is proved.

4 Shintani Descents

In this section we use the results and notation of §2.1 and §2.2. Just recall that \mathbf{G} is a simple group of type B_2 and that F is the generalized Frobenius map such that $\mathbf{G}^F = \text{Sz}(q)$ (where $q = 2^{2n+1}$). Recall furthermore that $\delta = 2$. We will first give Shintani descents between \tilde{G} and $\text{Sz}(q)$ and then obtain some results on the unipotent characters of $\text{Sz}(q)$.

4.1 Shintani correspondence

Proposition 4.1 *We have:*

- *If n is odd, then $N_{F/F^2} \text{Cl}(x_a x_b x_{a+b}) = \text{Cl}(x_a, \sigma)$.*
- *If n is even, then $N_{F/F^2} \text{Cl}(x_a x_b x_{a+b}) = \text{Cl}(x_a x_{a+b}, \sigma)$.*

Proof — We search for a $g \in \mathbf{G}$ such that $x_a = g^{-1}F(g)$. To this end suppose there exist $u, v, w, t \in \overline{\mathbb{F}}_2$ such that $g = x_a(u)x_b(v)x_{a+b}(w)x_{2a+b}(t)$. Then using the Chevalley relations of \mathbf{G} we immediately deduce:

$$\begin{aligned} F(g) &= x_a(v^{2^n})x_b(u^{2^{n+1}})x_{a+b}(t^{2^n} + v^{2^n}u^{2^{n+1}})x_{2a+b}(w^{2^{n+1}} + v^{2^{n+1}}u^{2^{n+1}}), \\ g x_a &= x_a(u+1)x_b(v)x_{a+b}(w+v)x_{2a+b}(t+v). \end{aligned}$$

By uniqueness of this decomposition we obtain:

$$\begin{cases} u + 1 & = v^{2^n} \\ v & = u^{2^{n+1}} \\ t^{2^n} + v^{2^n} u^{2^{n+1}} + w + v & = 0 \\ w^{2^{n+1}} + v^{2^{n+1}} u^{2^{n+1}} + t + v & = 0 \end{cases}$$

that is:

$$\begin{cases} u + 1 & = v^{2^n} \\ v & = u^{2^{n+1}} \\ t^{2^n} + w + uv & = 0 \\ w^{2^{n+1}} + t + u^2 v & = 0. \end{cases}$$

Suppose this system has a solution (u, v, w, t) . Then we have:

$$\begin{cases} u^q + u + 1 & = 0 \\ v^q + v + 1 & = 0 \\ w^q + w + v & = 0 \\ t^q + t + v & = 0. \end{cases}$$

Using these relations we find $gF^2(g^{-1}) = x_a x_b x_{a+b}(u + v + 1)x_{2a+b}(u^2 + v + 1)$. We now prove that this system has a solution. We write $h \in \overline{\mathbb{F}}_2$ for a root of $X^2 + X + 1$ and $k \in \overline{\mathbb{F}}_2$ for a root of $X^2 + X + h^2$ and we prove by induction on m that:

$$\begin{cases} h^{2^m} & = h + 1 & \text{if } m \text{ odd} \\ h^{2^m} & = h & \text{if } m \text{ is even} \end{cases}$$

and:

$$\begin{cases} k^{2^m} & = k & \text{if } m \equiv 0 \pmod{4} \\ k^{2^m} & = k + h^2 & \text{if } m \equiv 1 \pmod{4} \\ k^{2^m} & = k + 1 & \text{if } m \equiv 2 \pmod{4} \\ k^{2^m} & = k + h & \text{if } m \equiv 3 \pmod{4}. \end{cases}$$

We now study each of these cases:

- Suppose $n \equiv 1 \pmod{4}$. In this case n is odd. We set $v = u = h$ and $t = w = k$. We get $t^{2^n} + t + uv = 0$ and $t^{2^{n+1}} + t + u^2 v = 0$. The element $g = x_a(h)x_b(h)x_{a+b}(k)x_{2a+b}(k)$ is a solution and we have $gF^2(g^{-1}) = x_a x_b x_{a+b}$.
- Suppose $n \equiv 3 \pmod{4}$. In this case n is odd. We set $v = u = h$ and $t = w = h + k$. We obtain $t^{2^n} + w + uv = 0$ and $w^{2^{n+1}} + t + u^2 v = 0$. The element $g = x_a(h)x_b(h)x_{a+b}(h + k)x_{2a+b}(h + k)$ is a solution and $gF^2(g^{-1}) = x_a x_b x_{a+b}$.
- Suppose $n \equiv 0 \pmod{4}$. In this case n is even. We set $u = h$, $v = u + 1$, $t = k$ and $w = k + 1$. The element $g = x_a(h)x_b(h + 1)x_{a+b}(k + 1)x_{2a+b}(k)$ is a solution and we have $gF^2(g^{-1}) = x_a x_b x_{2a+b}$.

- Suppose $n = 2 \pmod{4}$. In this case n is even. We put $u = h, v = u+1, t = k$ and $w = k$. Then the element $g = x_a(h)x_b(h+1)x_{a+b}(k)x_{2a+b}(k)$ is a solution and we have $gF^2(g^{-1}) = x_ax_bx_{2a+b}$.

Using the definition of the Shintani correspondence (see §2.1.3), the claim is proved. \square

4.2 Shintani descents of unipotent characters

We fix i a primitive fourth root of unity. We recall that $\rho_0 = x_ax_{a+b}$ and that St is the Steinberg character of $\text{Sz}(q)$. We denote by \mathcal{W} the cuspidal unipotent character of degree $\theta(q-1)$ such that $\mathcal{W}(\rho_0) = \theta i$. We write $1_{\tilde{G}}, \tilde{\theta}_1, \tilde{\theta}_4$ and $\tilde{\theta}_5$ for the extensions of the unipotent characters of G as above.

Theorem 4.1 *We have $\text{Sh}_{F^2/F} 1_{\tilde{G}} = 1_{\text{Sz}(q)}$ and $\text{Sh}_{F^2/F} \tilde{\theta}_4 = \text{St}$. Setting $\zeta_0 = \frac{\sqrt{2}}{2}(-1-i)$ we have:*

- If n is even, then:

$$\begin{aligned} \text{Sh}_{F^2/F} \tilde{\theta}_5 &= -\frac{\zeta_0}{\sqrt{2}} \mathcal{W} - \frac{\overline{\zeta_0}}{\sqrt{2}} \overline{\mathcal{W}} \\ \text{Sh}_{F^2/F} \tilde{\theta}_1 &= -\frac{\zeta_0}{\sqrt{2}} \mathcal{W} - \frac{\overline{\zeta_0}}{\sqrt{2}} \overline{\mathcal{W}} \end{aligned}$$

- If n is odd, then:

$$\begin{aligned} \text{Sh}_{F^2/F} \tilde{\theta}_1 &= -\frac{\zeta_0}{\sqrt{2}} \mathcal{W} - \frac{\overline{\zeta_0}}{\sqrt{2}} \overline{\mathcal{W}} \\ \text{Sh}_{F^2/F} \tilde{\theta}_5 &= -\frac{\zeta_0}{\sqrt{2}} \mathcal{W} - \frac{\overline{\zeta_0}}{\sqrt{2}} \overline{\mathcal{W}} \end{aligned}$$

Proof — We say that a class of $\text{Sz}(q)$ is of type π_i ($i \in \{0, 1, 2\}$) if there exists some $k \in E_i$ such that π_i^k is a representative of the class. Similarly we say that a class of \tilde{G} is of type π_i ($i \in \{0, 1, 2\}$) if there exists some $k \in E_i$ such that (π_i^k, σ) is a representative of the class. Using Relation 1 in §2.1.3 and Theorem 3.1, we see that the classes of $\text{Sz}(q)$ of type π_0, π_1 and π_2 are sent by N_{F/F^2} to classes of \tilde{G} of type $(\pi_0, \sigma), (\pi_1, \sigma)$ and (π_2, σ) respectively. Since $1_{\tilde{G}}, \tilde{\theta}_1, \tilde{\theta}_4$ and $\tilde{\theta}_5$ are constant on these classes, we do not explicitly need to know the correspondence of these classes to compute the Shintani descents of these characters. Using a similar argument we obtain that $N_{F/F^2} \text{Cl}(1) = \text{Cl}(1, \sigma)$ and $N_{F/F^2} \text{Cl}(x_{a+b}x_{2a+b}) = \text{Cl}(x_{a+b}, \sigma)$. We now use Proposition 4.1

- Suppose that n is odd. Then we get

$$N_{F/F^2} \text{Cl}(\rho_0) = \text{Cl}(x_a, \sigma) \quad \text{and} \quad N_{F/F^2} \text{Cl}(\rho_0^{-1}) = \text{Cl}(x_ax_{a+b}, \sigma).$$

Using the values of outer characters obtained in Theorem 1.1 we have:

	1	ρ_0	σ_0	ρ_0^{-1}	π_0^i	π_1^j	π_2^k
$\text{Sh}_{F^2/F} 1_{\tilde{G}}$	1	1	1	1	1	1	1
$\text{Sh}_{F^2/F} \tilde{\theta}_4$	q^2	0	0	0	1	-1	-1
$\text{Sh}_{F^2/F} \tilde{\theta}_1$	$\theta(q-1)$	θ	$-\theta$	$-\theta$	0	1	-1
$\text{Sh}_{F^2/F} \tilde{\theta}_5$	$\theta(q-1)$	$-\theta$	$-\theta$	θ	0	1	-1

This yields $\text{Sh}_{F^2/F} 1_{\tilde{G}} = 1$ and $\text{Sh}_{F^2/F} \tilde{\theta}_4 = \text{St}$. Moreover, using the character Table of $\text{Sz}(q)$ we compute:

$$\begin{aligned}
\langle \text{Sh}_{F^2/F} \tilde{\theta}_1, 1 \rangle_{\text{Sz}} &= 0 & \langle \text{Sh}_{F^2/F} \tilde{\theta}_5, 1 \rangle_{\text{Sz}} &= 0 \\
\langle \text{Sh}_{F^2/F} \tilde{\theta}_1, \text{St} \rangle_{\text{Sz}} &= 0 & \langle \text{Sh}_{F^2/F} \tilde{\theta}_5, \text{St} \rangle_{\text{Sz}} &= 0 \\
\langle \text{Sh}_{F^2/F} \tilde{\theta}_1, \mathcal{W} \rangle_{\text{Sz}} &= -\zeta_0 \sqrt{2}/2 & \langle \text{Sh}_{F^2/F} \tilde{\theta}_5, \mathcal{W} \rangle_{\text{Sz}} &= -\overline{\zeta}_0 \sqrt{2}/2 \\
\langle \text{Sh}_{F^2/F} \tilde{\theta}_1, \overline{\mathcal{W}} \rangle_{\text{Sz}} &= -\overline{\zeta}_0 \sqrt{2}/2 & \langle \text{Sh}_{F^2/F} \tilde{\theta}_5, \overline{\mathcal{W}} \rangle_{\text{Sz}} &= -\zeta_0 \sqrt{2}/2
\end{aligned}$$

We thus deduce that:

$$\begin{aligned}
\text{Sh}_{F^2/F} \tilde{\theta}_1 &= -\zeta_0 \sqrt{2}/2 \mathcal{W} - \overline{\zeta}_0 \sqrt{2}/2 \overline{\mathcal{W}} \\
\text{Sh}_{F^2/F} \tilde{\theta}_5 &= -\overline{\zeta}_0 \sqrt{2}/2 \mathcal{W} - \zeta_0 \sqrt{2}/2 \overline{\mathcal{W}}
\end{aligned}$$

- If n is even, we proceed similarly using the identities $N_{F/F^2} \text{Cl}(\rho_0) = \text{Cl}(x_a x_{a+b}, \sigma)$ and $N_{F/F^2} \text{Cl}(\rho_0^{-1}) = \text{Cl}(x_a, \sigma)$.

□

4.3 Roots associated to the unipotent characters of $\text{Sz}(q)$

We denote by $\omega_{\mathcal{W}}$ and $\omega_{\overline{\mathcal{W}}}$ the roots of unity associated to \mathcal{W} and $\overline{\mathcal{W}}$ as in §2.1.1. In [7] §7.4 G. Lusztig shows that $\{\omega_{\mathcal{W}}, \omega_{\overline{\mathcal{W}}}\} = \{\zeta_0, \overline{\zeta}_0\}$. We now make this result more precise.

Theorem 4.2 *The roots associated to \mathcal{W} and $\overline{\mathcal{W}}$ are:*

	\mathcal{W}	$\overline{\mathcal{W}}$
$n \text{ odd}$	ζ_0	$\overline{\zeta}_0$
$n \text{ even}$	$\overline{\zeta}_0$	ζ_0

Proof — To compute the almost characters of $\text{Sz}(q)$ we need to know the generalized Deligne-Lusztig characters R_w . The F -classes of W have the representatives 1, w_a and $w_a w_b w_a$. The Suzuki group with parameter q therefore has three Deligne-Lusztig characters with the degrees:

$$R_1(1) = q^2 + 1, \quad R_{w_a}(1) = (q-1)(q-r+1), \quad R_{w_a w_b w_a}(1) = (q-1)(q+r+1).$$

These characters are explicitly computed in [10]. We recall that (see [10] Prop. 4.6.7):

$$\begin{aligned} R_1 &= 1 + \text{St} \\ R_{w_a} &= 1 - \mathcal{W} - \overline{\mathcal{W}} - \text{St} \\ R_{w_a w_b w_a} &= 1 + \mathcal{W} + \overline{\mathcal{W}} - \text{St}. \end{aligned}$$

The Weyl group W has three F -stable characters denoted by ρ_1 , ρ_2 and ρ_3 . Let $\tilde{\rho}_1$, $\tilde{\rho}_2$ and $\tilde{\rho}_3$ be their extensions to $W \rtimes \langle F \rangle$ such that:

	$(1, F)$	(w_a, F)	$(w_a w_b w_a, F)$
$\tilde{\rho}_1$	1	1	1
$\tilde{\rho}_2$	1	-1	-1
$\tilde{\rho}_3$	0	$-\sqrt{2}$	$\sqrt{2}$

The three almost characters of $\text{Sz}(q)$ corresponding to these extensions are (§2.1.2):

$$\begin{aligned} \mathcal{R}_{\tilde{\rho}_1} &= 1_{\text{Sz}(q)} \\ \mathcal{R}_{\tilde{\rho}_2} &= \text{St} \\ \mathcal{R}_{\tilde{\rho}_3} &= \frac{\sqrt{2}}{2}(\mathcal{W} + \overline{\mathcal{W}}). \end{aligned}$$

Therefore we have:

$$\langle \mathcal{R}_{\tilde{\rho}_3}, \mathcal{W} \rangle_{\text{Sz}} = \sqrt{2}/2 \quad \text{and} \quad \langle \mathcal{R}_{\tilde{\rho}_3}, \overline{\mathcal{W}} \rangle_{\text{Sz}} = \sqrt{2}/2.$$

Moreover θ_1 is in the principal series of G . Using Theorem 2.1 we deduce that:

$$\sqrt{2} \langle \text{Sh}_{F^2/F} \tilde{\theta}_1, \mathcal{W} \rangle_{\text{Sz}} = \pm \omega_{\mathcal{W}}.$$

The sign is due to the fact that $\text{Sh}_{F^2/F} \tilde{\theta}_1 = \pm \text{Sh}_{F^2/F} \chi_{\tilde{\rho}_3}$. Since $\omega_{\mathcal{W}}$ is either ζ_0 or $\overline{\zeta}_0$ and using that $\overline{\zeta}_0 \neq -\zeta_0$, we can obtain the root. Indeed

- Either $\sqrt{2} \langle \text{Sh}_{F^2/F} \tilde{\theta}_1, \mathcal{W} \rangle$ is ζ_0 or $\overline{\zeta}_0$ and in this case $\omega_{\mathcal{W}} = \sqrt{2} \langle \text{Sh}_{F^2/F} \tilde{\theta}_1, \mathcal{W} \rangle$.
- Or $\sqrt{2} \langle \text{Sh}_{F^2/F} \tilde{\theta}_1, \mathcal{W} \rangle$ is neither ζ_0 nor $\overline{\zeta}_0$ and then $\omega_{\mathcal{W}} = -\sqrt{2} \langle \text{Sh}_{F^2/F} \tilde{\theta}_1, \mathcal{W} \rangle$.

Using 4.1 the claim is proved. \square

4.4 Fourier matrices

The unipotent characters of $\text{Sz}(q)$ are distributed in three families $\mathcal{F}_1 = \{1\}$, $\mathcal{F}_2 = \{\text{St}\}$ and $\mathcal{F}_3 = \{\mathcal{W}, \overline{\mathcal{W}}\}$.

Proposition 4.2 *The Fourier matrices M_i ($i = 1, 2, 3$) associated to the \mathcal{F}_i can be define up to a normalization by*

$$M_1 = M_2 = [1], \quad \text{and} \quad M_3 = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}.$$

Proof — Theorem 4.1 shows that the irreducible components of $\text{Sh}_{F^2/F} \tilde{\theta}_1$ and of $\text{Sh}_{F^2/F} \tilde{\theta}_5$ are the elements of the family \mathcal{F}_3 . On the other hand we deduce from Theorem 4.1 that:

$$\text{If } n \text{ even: } i \text{Sh}_{F^2/F} \tilde{\theta}_5 = -\overline{\zeta_0} \sqrt{2}/2\mathcal{W} + \zeta_0 \sqrt{2}/2\overline{\mathcal{W}}$$

$$\text{If } n \text{ is odd: } i \text{Sh}_{F^2/F} \tilde{\theta}_5 = \zeta_0 \sqrt{2}/2\mathcal{W} - \overline{\zeta_0} \sqrt{2}/2\overline{\mathcal{W}}.$$

Using Conjecture 2.1 we can define the Fourier matrix associated to \mathcal{F}_3 as claimed. □

Appendix

Let n be a non-negative integer and write $q = 2^{2n+1}$. The character table of $B_2(q)$ is given in [6] p. 93. In Table 10 we recall the conjugacy classes of $B_2(q)$.

Remark 4.1 *The two classes of $B_2(q)$ whose centralizers are of order $2q^2$ are $\text{Cl}(x_a x_b)$ and $\text{Cl}(x_a x_b x_{a+b})$. Indeed suppose there exists $\alpha \in \mathbb{F}_q$ such that $P(\alpha) = 0$, where $P = X^2 + X + 1$. We can suppose that $\alpha \notin \{0, 1\}$ because 0 and 1 are no roots of P . Moreover $1 - \alpha^3 = (1 - \alpha)(\alpha^2 + \alpha + 1) = 0$ because α is a root of P . We deduce that the order of α divides 3. Furthermore $\alpha \neq 1$, thus the order of α is 3. It follows that 3 divides $(q - 1)$, that is $q - 1 = 0 \pmod{3}$ which is false. Thus P is irreducible.*

We set $\alpha_i = \gamma_0^i + \gamma_0^{-i}$ and $\beta_i = \nu_0^k + \nu_0^{-k}$. In Table 11, we recall the irreducible characters that we need in this work and correct some errors of [6].

We recall the character table of $\text{Sz}(q)$ which is computed in [14] in Table 12. Here we set $\theta = 2^n$.

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	$\text{Ind}_{U_0}^B \phi$
A_1	$\frac{1}{4}q^2(q-1)^2\phi(1)$
A_2	$\frac{1}{4}q^2(q-1) \sum_{u \in K^\times} \phi(x_{2a+b}(u))$
A_{31}	$\frac{1}{4}q^2(q-1) \sum_{u \in K^\times} x_{a+b}(u)$
A_{32}	$\frac{q^2}{4} \sum_{u, v \in K^\times} \phi(x_{a+b}(u)x_{2a+b}(v))$
A_{41}	$\frac{1}{2}q(q-1) + \sum_{u \neq u+1} \phi(x_b x_{a+b}(u)x_{2a+b}(u^2))$
A_{42}	$\frac{q}{2} \sum_{u \in K^\times} \phi(x_b x_{a+b}(u)) + \sum_{u \neq 0, 1} \phi(x_b x_{2a+b}(u)) + \sum_{u \neq v^2} \phi(x_b x_{a+b}(u)x_{2a+b}(v))$
A_{51}	$\frac{1}{2}q(q-1) \sum_{u \neq u+1} \phi(x_a x_{a+b}(u)x_{2a+b}(u))$
A_{52}	$\frac{q}{2} \sum_{u \in K^\times} \phi(x_a x_{a+b}(u)) + \sum_{u \neq 0, 1} \phi(x_a x_{2a+b}(u)) + \sum_{u \neq v} \phi(x_b x_{a+b}(u)x_{2a+b}(v))$
A_{61}	$\sum_{\lambda(u)=1} \phi(x_a x_b x_{a+b}(u)) + \sum_{\lambda(u)=1, u \neq 0} \phi(x_a x_b x_{2a+b}(u))$ $+ \sum_{\lambda(u+v)=1} \phi(x_a x_b x_{a+b}(u)x_{2a+b}(v))$
A_{62}	$\sum_{\lambda(u)=-1} \phi(x_a x_b x_{a+b}(u)) + \sum_{\lambda(u)=-1, u \neq 1} \phi(x_a x_b x_{2a+b}(u))$ $+ \sum_{\lambda(u+v)=-1} \phi(x_a x_b x_{a+b}(u)x_{2a+b}(v))$

Table 6: Induction formula from U_0 to B

	$(1, \sigma)$	(x_a, σ)	(x_{a+b}, σ)	$(x_a x_{a+b}, \sigma)$
$\text{Ind}_{U_0}^{\tilde{B}} \phi$	$\frac{1}{2}q(q-1)\phi(1, \sigma)$	$\sum_{\lambda(u)=1} \phi(x_a x_{a+b}(u), \sigma)$	$\frac{q}{2} \sum_{u \neq 0} \phi(x_{a+b}(u), \sigma)$	$\sum_{\lambda(u)=-1} \phi(x_a x_{a+b}(u), \sigma)$

Table 7: Outer values of the induction formula.

	A_1	A_2	A_{31}	A_{32}	A_{41}
$\text{Ind}_{U_0}^B \lambda(1, 1)$	$\frac{1}{4}q^2(q-1)^2$	$-\frac{1}{4}q^2(q-1)$	$-\frac{1}{4}q^2(q-1)$	$\frac{q^2}{4}$	$-\frac{1}{4}q^2(q-1)$
	A_{42}	A_{51}	A_{52}	A_{61}	A_{62}
	$\frac{q^2}{4}$	$-\frac{1}{4}q^2(q-1)$	$\frac{q^2}{4}$	$\frac{q^2}{4}$	$\frac{-q^2}{4}$

	$(1, \sigma)$	(x_a, σ)	(x_{a+b}, σ)	$(x_a x_{a+b}, \sigma)$
$\text{Ind}_{U_0}^{\tilde{B}} \tilde{\lambda}(1, 1)$	$\frac{1}{2}q(q-1)$	$-\frac{q}{2}$	$-\frac{q}{2}$	$\frac{q}{2}$

Table 8: Values of $\text{Ind}_{U_0}^B \lambda(1, 1)$ and outer values of $\text{Ind}_{U_0}^{\tilde{B}} \tilde{\lambda}(1, 1)$

		$(1, \sigma)$	(x_a, σ)	(x_{a+b}, σ)	$(x_a x_{a+b}, \sigma)$	(π_0, σ)	(π_1, σ)	(π_2, σ)
$ C_{\tilde{G}} $		$2q^2(q-1)(q^2+1)$	$4q$	$2q^2$	$4q$	$2(q-1)$	$2(q+2\theta+1)$	$2(q-2\theta+1)$
1	1	1	1	1	1	1	1	1
$\tilde{\theta}_4$	1	q^2	0	0	0	1	-1	-1
$\tilde{\theta}_1$	1	$\theta(q-1)$	θ	$-\theta$	$-\theta$	0	1	-1
$\tilde{\theta}_5$	1	$\theta(q-1)$	$-\theta$	$-\theta$	θ	0	1	-1
$\tilde{\chi}_{\pi_0}(i)$	$i \in E_0$	q^2+1	1	1	1	$\varepsilon_0^i(\pi_0)$	0	0
$\tilde{\chi}_{\pi_1}(j)$	$j \in E_1$	$(q-1)(q-2\theta+1)$	-1	$2\theta-1$	-1	0	$-\varepsilon_1^j(\pi_1)$	0
$\tilde{\chi}_{\pi_2}(k)$	$k \in E_2$	$(q-1)(q+2\theta+1)$	-1	$-2\theta-1$	-1	0	0	$-\varepsilon_2^k(\pi_2)$

Table 9: Values of the outer characters $B_2(q) \rtimes \langle \sigma \rangle$ on outer classes

Notation	Representatives	Number	Centralizer's order
A_1	$h(1, 1)$	1	$q^4(q^2 - 1)(q^4 - 1)$
A_2	x_{2a+b}	1	$q^4(q^2 - 1)$
A_{31}	x_{a+b}	1	$q^4(q^2 - 1)$
A_{32}	$x_{a+b}x_{2a+b}$	1	q^4
A_{41}	x_ax_b	1	$2q^2$
A_{42}	$x_ax_bx_{2a+b}$	1	$2q^2$
$B_1(i, j)$	$h(\gamma^i, \gamma^j)$	$\frac{1}{8}(q-2)(q-4)$	$(q-1)^2$
$B_2(i)$	$h(\tau'^i, \tau'^{qi})$	$\frac{1}{4}q(q-2)$	$q^2 - 1$
$B_3(i, j)$	$h(\gamma^i, \nu^j)$	$\frac{1}{4}q(q-2)$	$q^2 - 1$
$B_4(i, j)$	$h(\nu^i, \nu^j)$	$\frac{1}{8}q(q-2)$	$(q+1)^2$
$B_5(i)$	$h(\tau^i, \tau^{qi})$	$\frac{1}{4}q^2$	$q^2 + 1$
$C_1(i)$	$h(1, \gamma^i)$	$\frac{1}{2}(q-2)$	$q(q-1)(q^2-1)$
$C_2(i)$	$h(\gamma^i, \gamma^{-i})$	$\frac{1}{2}(q-2)$	$q(q-1)(q^2-1)$
$C_3(i)$	$h(1, \nu^i)$	$\frac{1}{2}q$	$q(q+1)(q^2-1)$
$C_4(i)$	$h(\nu^i, \nu^{-i})$	$\frac{1}{2}q$	$q(q+1)(q^2-1)$
$C_1(i)$	$h(1, \gamma^i)x_{2a+b}$	$\frac{1}{2}(q-2)$	$q(q-1)$
$C_2(i)$	$h(\gamma^i, \gamma^{-i})x_{a+b}$	$\frac{1}{2}(q-2)$	$q(q-1)$
$C_3(i)$	$h(1, \nu^i)x_{2a+b}$	$\frac{1}{2}q$	$q(q+1)$
$C_4(i)$	$h(\nu^i, \nu^{-i})x_{a+b}$	$\frac{1}{2}q$	$q(q+1)$

Table 10: Conjugacy classes of $B_2(q)$

	A_1	A_2	A_{31}	A_{32}	A_{41}	A_{42}	$B_1(i, j)$	$B_2(i)$	$B_3(i, j)$	$B_4(i, j)$
θ_1	$q(q+1)^2/2$	$q(q+1)/2$	$q(q+1)/2$	$q/2$	$q/2$	$-q/2$	2	0	0	0
θ_4	q^4	0	0	0	0	0	1	-1	-1	1
θ_5	$q(q-1)^2/2$	$-q(q-1)/2$	$-q(q-1)/2$	$q/2$	$q/2$	$-q/2$	0	0	0	-2
$\chi_1(k, l)$	$(q+1)^2(q^2+1)$	$(q+1)^2$	$(q+1)^2$	$2q+1$	1	1	$\alpha_{ik}\alpha_{jl} + \alpha_{il}\alpha_{jk}$	0	0	0
$\chi_4(k, l)$	$(q-1)^2(q^2+1)$	$(q-1)^2$	$(q-1)^2$	$-(2q-1)$	1	1	0	0	0	$\beta_{ik}\beta_{jl} + \beta_{il}\beta_{jk}$
$\chi_5(k)$	$(q^2-1)^2$	$-(q^2-1)$	$-(q^2-1)$	1	1	1	0	0	0	0

	$B_5(i)$	$C_1(i)$	$C_2(i)$	$C_3(i)$	$C_4(i)$	$D_1(i)$	$D_2(i)$	$D_3(i)$	$D_4(i)$
θ_1	-1	$q+1$	$q+1$	0	0	1	1	0	0
θ_4	1	q	q	$-q$	$-q$	0	0	-1	-1
θ_5	1	0	0	$q-1$	$q-1$	0	0	-1	-1
$\chi_1(k, l)$	0	$(q+1)(\alpha_{ik} + \alpha_{il})$	$(q+1)\alpha_{ik}\alpha_{il}$	0	0	$\alpha_{ik} + \alpha_{il}$	$\alpha_{ik}\alpha_{il}$	0	0
$\chi_4(k, l)$	0	0	0	$-(q-1)(\beta_{ik} + \beta_{il})$	$-(q-1)\beta_{ik}\beta_{il}$	0	0	$\beta_{ik} + \beta_{il}$	$\beta_{ik}\beta_{il}$
$\chi_5(k)$	$\tau^{ik} + \tau^{-ik} + \tau^{ikq} + \tau^{-ikq}$	0	0	0	0	0	0	0	0

Table 11: Character table of $B_2(q)$

	1	σ_0	ρ_0	ρ_0^{-1}	π_0^l	π_1^l	π_2^l
$1_{\text{Sz}(q)}$	1	1	1	1	1	1	1
St	q^2	0	0	0	1	-1	-1
\mathcal{W}	$\theta(q-1)$	$-\theta$	$\theta\sqrt{-1}$	$-\theta\sqrt{-1}$	0	1	-1
$\overline{\mathcal{W}}$	$\theta(q-1)$	$-\theta$	$-\theta\sqrt{-1}$	$\theta\sqrt{-1}$	0	1	-1
X_i	q^2+1	1	1	1	$\varepsilon_0^i(\pi_0^l)$	0	0
Y_j	$(q-2\theta+1)(q-1)$	$2\theta-1$	-1	-1	0	$-\varepsilon_1^j(\pi_1^l)$	0
Z_k	$(q+2\theta+1)(q-1)$	$-2\theta-1$	-1	-1	0	0	$-\varepsilon_2^k(\pi_2^l)$

Table 12: Character table of $\text{Sz}(q)$